

HYDROMAGNETIC INTERACTION IN A FLUID DIELECTRIC

(Letter to the Editor)

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(Received 30 January, 1979)

Abstract. It is shown that a dielectric fluid is capable of hydromagnetic interaction, much like a conducting fluid. The physical principle of this interaction is outlined and the criteria for dominance of the interaction are derived.

The subject of magnetohydrodynamics deals with the behavior of a conducting fluid permeated by a magnetic field and constitutes a very fruitful and vigorous branch of study in physics. We suggest here an almost parallel theoretical development dealing with the dynamic behavior of a dielectric fluid having a finite molecular polarizability, when permeated by a magnetic field.

Consider a fluid dielectric with a polarizability χ permeated by a magnetic field \mathbf{B} . If the fluid has a bulk velocity \mathbf{u} (assumed nonrelativistic; i.e., $u \ll c$, the velocity of light), then an electric field \mathbf{E} measured in the laboratory (or rest) frame translates to a field \mathbf{E}' in the body of the moving fluid, where (see, e.g., Stratton, 1941)

$$\mathbf{E}' = \mathbf{E} + \mathbf{u} \times \mathbf{B}. \quad (1)$$

In a time-varying situation, the field \mathbf{E}' produces a polarization current density \mathbf{J}' in the moving fluid, given by

$$\mathbf{J}' = \chi' \epsilon_0 \dot{\mathbf{E}}', \quad (2)$$

where χ' refers to a frame moving with the fluid, and ϵ_0 is the permittivity of free space. If there is a space charge density ϱ'_c in the fluid, then the net electric current in the rest frame is (*op. cit.*)

$$\mathbf{J} = \mathbf{J}' + \varrho'_c \mathbf{u}. \quad (3)$$

At nonrelativistic velocities we may assume that $\chi' \equiv \chi$ and $\varrho'_c \equiv \varrho_c$, the space charge density in the rest frame. We now have

$$\mathbf{J} = \chi \epsilon_0 \left[\dot{\mathbf{E}} + \frac{\partial}{\partial t} (\mathbf{u} \times \mathbf{B}) \right] + \varrho_c \mathbf{u}, \quad (4)$$

where the last term is the convection current. The above equation may be viewed as a

simple dielectric analog of the generalized Ohm's law of magnetohydrodynamics. To this, we add Newton's law

$$\frac{\partial}{\partial t} (\varrho \mathbf{u}) = \mathbf{J} \times \mathbf{B} + \varrho_c \mathbf{E} \quad (5)$$

(where ϱ is the fluid mass density, and where nonelectromagnetic forces are neglected), and the four Maxwell equations

$$\nabla \cdot \mathbf{B} = 0, \quad (6)$$

$$\nabla \cdot \chi \epsilon_0 \mathbf{E} = -\varrho_c, \quad (7)$$

$$\nabla \times \mathbf{B} = \mu \mathbf{J} + \mu \epsilon_0 \dot{\mathbf{E}}, \quad (8)$$

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}; \quad (9)$$

where ϱ_c is defined now as the density of *polarization* charges, and μ is the magnetic permeability of the fluid. Equation (7) – a modified form of Maxwell's equation – is written in terms of the polarization charge since this is the only space charge that can arise in the body of the moving fluid, any free charges arising on a bounding conductor. In Equation (8), any role of a magnetization current is neglected. Note that the conventional dielectric permittivity of the fluid is

$$\epsilon = (1 + \chi) \epsilon_0, \quad (10)$$

and is assumed for the present discussion to be a scalar quantity. The set of Equations (4)–(10) forms the complete basis of the discussion of the ideal hydromagnetic interaction in a fluid dielectric. Note that even though the quantity $\mathbf{J} \cdot \mathbf{E}$ may be nonzero, there is no energy dissipation in the dielectric except when ϵ has an imaginary component.

To develop our discussion somewhat further, we next assume that χ has the same value over the entire body of the fluid, and that it does not change with time. Using Equations (6), (8) and (9) in Equation (4), we obtain

$$\ddot{\mathbf{B}} = \frac{1}{\mu \epsilon} \nabla^2 \mathbf{B} + \frac{\chi}{1 + \chi} \frac{\partial}{\partial t} \nabla \times (\mathbf{u} \times \mathbf{B}) + \frac{1}{1 + \chi} \frac{1}{\epsilon_0} \nabla \times (\varrho_c \mathbf{u}). \quad (11)$$

Let the terms on the right-hand side be labeled T_1 , T_2 and T_3 , respectively. They have *a priori* magnitudes

$$T_1 \sim \frac{B}{\mu \epsilon L^2}, \quad T_2 \sim \frac{\chi}{1 + \chi} \frac{uB}{\tau L}, \quad T_3 \sim \frac{\chi}{1 + \chi} \frac{uB}{\tau l},$$

where L is the characteristic dimension of the moving region, l is the length-scale for the divergence of \mathbf{E} , and τ is the characteristic time-scale of the problem.

If the first term on the right-hand side of Equation (11) dominates – i.e., if

$$T_2/T_1 = X = uL\chi\mu\varepsilon_0/\tau \ll 1$$

and

$$T_3/T_1 = Y = ul\chi\mu\varepsilon_0/\tau \ll 1,$$

then this equation reduces to

$$\ddot{\mathbf{B}} = c_d^2 \nabla^2 \mathbf{B}, \quad (12)$$

where $c_d = (\mu\varepsilon)^{-1/2}$ is the velocity of electromagnetic wave in the dielectric. This is a three-dimensional wave equation – indicating that the disturbance is purely electromagnetic in nature.

If the term T_2 dominates on the right-hand side of Equation (11) – i.e., if

$$X \gg 1 \quad \text{and} \quad l/L \gg 1,$$

then we have

$$\ddot{\mathbf{B}} = \frac{\chi}{1 + \chi} \frac{\partial}{\partial t} \nabla \times (\mathbf{u} \times \mathbf{B}). \quad (13)$$

Integrating once with respect to time, we have

$$\dot{\mathbf{B}} = \frac{\chi}{1 + \chi} \nabla \times (\mathbf{u} \times \mathbf{B}) + \mathbf{C}, \quad (14)$$

where the integration constant \mathbf{C} must be set equal to zero, since otherwise it indicates a monotonic change in \mathbf{B} with time even when $\mathbf{u} = 0$ everywhere. Under the condition $\chi \gg 1$, Equation (14) now reduces to

$$\dot{\mathbf{B}} = \nabla \times (\mathbf{u} \times \mathbf{B}), \quad (15)$$

a condition which is easily shown to indicate a state of frozen flow (see, e.g., Alfvén and Fälthammar, 1963): the magnetic flux through any contour attached to the moving fluid stays constant with time.

Note next that although the condition $X \gg 1$ is necessary for the *predominance* of hydromagnetic interaction, it is not sufficient since this condition can be satisfied even when the ambient magnetic field \mathbf{B}_0 (where $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$, \mathbf{b} being the motionally induced magnetic field) is infinitesimally small. In order for the interaction to be appreciable, we must additionally take into account Equation (5) which says that the electromagnetic force is comparable to the inertial force. Under the assumptions that the fluid is incompressible and that $X \gg 1$ and $l/L \gg 1$, we obtain from Equations (5) and (8) the relation

$$\frac{\rho u}{\tau} \approx \frac{Bb}{\mu L}. \quad (16)$$

The condition that the second term on the right-hand side of Equation (11) far exceeds the first term now becomes

$$\frac{uB}{\tau L} \gg \frac{b}{\mu e L^2}. \quad (17)$$

Combining Equation (16) with inequality (17), we have

$$Z = \frac{\epsilon}{\epsilon_{eq}} \gg 1 \quad (18)$$

as the necessary *and* sufficient criterion for the *dominance* of hydromagnetic interaction in a fluid dielectric (where $\epsilon_{eq} = \varrho/B^2$ is an *equivalent* permittivity). Strong magnetic field, small mass density and large χ favor this criterion. Examples of fluids with large values of χ are liquid H₂O, H₂O₂, D₂O, N₂H₄ and many organic compounds at room temperature, while H₂, He, etc., have appreciable χ at near-cryogenic temperatures. It may be noted incidentally that Equations (4)–(10) can be combined to yield a hydromagnetic wave behavior analogous to Alfvén waves of magnetohydrodynamics (see, e.g., Alfvén and Fälthammar, 1963). This wave behaviour has been reported elsewhere (De, 1979).

Finally, if the term T_3 in Equation (11) dominates over T_1 and T_2 – i.e., if

$$Y \gg 1 \quad \text{and} \quad L/l \gg 1,$$

– then we have

$$\ddot{\mathbf{B}} = \frac{1}{1 + \chi} \frac{1}{\epsilon_0} \nabla \times (\varrho_c \mathbf{u}), \quad (19)$$

indicating that the induced magnetic field is now perpendicular to the direction of fluid motion. There is no frozen flow; instead, the induced magnetic field \mathbf{b} may be parallel to the initial field \mathbf{B}_0 .

The Lunar and Planetary Institute is operated by the Universities Space Research Association under Contract No. NSR 09-051-001 with the National Aeronautics and Space Administration. This work constitutes the Lunar and Planetary Institute Contribution No. 356.

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